

Chaos control in traffic flow models

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Nowadays traffic flow problems has acquired interdisciplinary status due to the following reasons: first of all to investigate traffic flow models methods from different branches of science such as hydrodynamics, theory of magnetizm, cybernetics, etc. are applied; secondly results of investigations of traffic problems with different approaches could be adequate in various scientific directions, see, e.g. refs.1-9.

Mean field approach is one of widely used approaches in traffic problems.(for some publications, see, ref.8-10 and references therein). It is well-known that nowadays cellular automaton (CA) models has extensive applications to the traffic flow models.In this paper we dwell on just two models from the mean field theory:one- and two-dimensional systems.First we will consider one dimensional model.

Recently, the authors of ref.8 have presented microscopic derivations of mean field theories for CA models of traffic flow in one dimension.They established the following mapping between the average velocities of cars v at times $t+1$ and t :

$$v(t+1) = (2-f)v(t) - (1-p)^{-1}v^2(t) + p(1-p)^{-1}v^3(t), \quad (1)$$

where, p is the car density; f is the quantity responsible for the random delay due to the say, different driving habits and road conditions. So one has discrete dynamical nonlinear system, which exhibit rich dynamical behavior(see below after the presentation of two-dimensional traffic model)

Not so long ago Biham et al.(ref.2) (below simply BML) introduced a simple two dimensional (2D) CA model with traffic lights and studied the average velocity of cars as a function of their density.In that model, cars moving from west to east attempt to move in odd time steps and cars from south to north -in even time steps.There are three possible states on the square lattice:(i) occupied by an eastbound car; (ii)occupied by a northbound car; (iii)vacant. In ref.8 it was underlined that the divison of time into odd and even time steps simulates the effect of of traffic lights.The main result of BML:the average velocity in the long time limit vanishes when the density of cars p is higher than a critical value p_c .Below p_c , the traffic is in a moving phase, while above p_c it is in a jamming phase.Numerous improvements of the basic BML model taking into account the effects of factors such as overpasses, faulty traffic lights, asymmetric distribution of cars in a homogenous lattice, and traffic accidents,(see ref.8 and refereneces therein.)has been carried out. In the very recent paper (ref.8) an improved mean field theory for 2D traffic flow with a fraction c of overpass sites and with possible asymmetry in the distributions of cars in the two different directions is studied. The model in fact is the improved version of Nagatani model,

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ref.11. The overpass sites can be occupied simultaneously by an eastbound car and a northbound car, thus modelling the two-level overpasses in modern road systems in cities. Nagatani model deals with the isotropic distribution of cars with different overpass sites. It has been shown that the addition of overpasses enhances both the average speed of the traffic and critical density of cars. However the Nagatani model has some shortcomings in the sense that blockage of cars due to cars moving in the same direction is not taken into account properly, which led to too low estimation of the concentration of overpasses for the transition from jamming to moving phase at $p = 1$ and too high estimate of the critical car density at $c = 0$. Although in this paper we will deal with the isotropic distributions of cars for the sake of completeness we first write the system of equations with asymmetry. Let p_x and p_y to be the density of cars in the x (eastbound) and y (northbound) directions respectively. Also let v_x and v_y to be the average speeds of cars in the same directions. Let also c to be a fraction of overpass sites. Then according to ref.8 these quantities are related through the following nonlinear dynamical equations:

$$\begin{aligned} v_x &= 1 - (1 - c)(p_y v_y^{-1} + p_x(v_x^{-1} - 1)) \\ v_y &= 1 - (1 - c)(p_x v_x^{-1} + p_y(v_y^{-1} - 1)), \end{aligned} \quad (2)$$

From the mathematical point of view the system (2) also is the nonlinear dynamical system.

It is well-known that some dynamical systems depending on the value of systems' parameters exhibit unpredictable, chaotic behaviour, refs.10-15. The seminal papers (refs.12-13) induced avalanche of research works in the theory of control of chaos in synergetics. Chaos synchronization in dynamical systems is one of such ways of controlling chaos. In the spirit of refs.12-13 by synchronization of two systems we mean that the trajectories of one of the systems will converge to the same values as the other and they will remain in step with each other. For the chaotic systems synchronization is performed by the linking of chaotic systems with a common signal or signals (the so-called drivers)

According to refs.12-13 in the above mentioned way of chaos control one or some of these state variables can be used as an input to drive a subsystem that is a replica of part of the original system. In refs.12-13 it has been shown that if all the Lyapunov exponents (or the largest Lyapunov exponent) or the real parts of these exponents for the subsystem are negative then the subsystem synchronizes to the chaotic evolution of original system. If the largest subsystem Lyapunov exponent is not negative then as it has been proved in ref.18 synchronism is also possible. In this case a nonreplica system constructed according to ref.18 is used instead of replica subsystem.

The interest to the chaos synchronization in part is due to the application of this phenomenon in secure communications, in modeling of brain activity and recognition processes, etc, see, references in ref. 17). Also we should mention that this method of chaos control may result in the improved performance (according to some criterion) of chaotic systems (see, e.g.ref.17).

In this paper for the first time (to our knowledge) we report on the possible chaos control in the traffic flow models.

We will act within the algorithm proposed in ref.18.

Our paper is dedicated to the study of the stabilization of unstable behaviors in one dimensional and

application of both replica and nonreplica approaches to chaos control to the 2D traffic model. First we will investigate the one dimensional model. The stationary values of average velocity can be easily calculated from the equation (1). These values are:

$$v_1^{st} = 0, v_2^{st} = \frac{1 - (1 - 4(1 - f)p(1 - p))^{\frac{1}{2}}}{2p}, \quad (3)$$

The stability analysis of this stationary states show that :the v_1^{st} is always unstable, except for $p = 1$. Indeed, for this state

$$\left| \frac{v(t+1)}{v(t)} \right| = |2 - f| > 1, \quad (4)$$

As f changes between zero and unity. The instability condition for the second stationary state is:

$$|(1 - p)^{-1} \frac{3(1 - 4(1 - f)p(1 - p)) - 1 - 2(1 - 4(1 - f)p(1 - p))^{\frac{1}{2}}}{4p} + 2 - f| > 1, \quad (5)$$

As the stability analysis show in general we have stable and unstable states depending on the value of p, f . As a rule the unstable states are discarded as unphysical ones. But nowadays due to the success of chaos control theory it is possible to stabilize the unstable fixed points or periodic orbits. Below in dealing with the control of instability of fixed points in one dimensional map we will follow the so-called proportional feedback method described in ref.19, which is map based variation of a method proposed in ref.12. Following the method presented in ref.19 first we linearize the one dimensional map the one dimensional map (1) in the vicinity of the fixed points (or stationary states v^{st}):

$$v(t+1) = h(v(t) - v^{st}) + v^{st}, \quad (6)$$

where $|h| > 1$ is the slope of the map at v^{st} . In the mapping (1) we have only one parameter f by changing which one can stabilize the unstable fixed points. The positive answer to this problem vindicates the intuition that by improving driving habit, road conditions one can solve some traffic difficulties. Of course, theoretically the regularization of traffic flow also could be achieved by manipulation with p . Having this in mind let us denote the parameters as $m = (f, p)$. Now suppose that we change this parameter m by small amount δm to move the unstable fixed point v^{st} without significant changing of the slope of the map (1) h . In other words

$$v_{t+1}(m + \delta m) = h(v_t - v^{st}(m + \delta m)) + v^{st}(m + \delta m), \quad (7)$$

(in order to avoid confusion ,where necessary we write t as a subscript) where

$$v^{st}(m + \delta m) = \delta m \frac{dv^{st}}{dm} + v^{st}, \quad (8)$$

Now suppose that $v_t = v^{st}(m) + \delta_1 v$, where the second term in the right-hand side of this equality is much smaller than the first one. If at this moment m is changed to $m + \delta m$ such that $v_{t+1}(m + \delta m) = v^{st}(m)$,

the system state is directed to the original unstable fixed point upon the next iteration. If m is then switched back to its original value, the system would remain at v^{st} indefinitely. The necessary variations of m can easily be determined by the formula

$$\delta m = \frac{h}{(h-1)\frac{dv^{st}}{dm}} \delta_1 v = \frac{\delta_1 v}{g}, \quad (9)$$

One can see easily the necessary changing of parameters to stabilise the unstable fixed points is proportional to the deviation of v from the fixed point (or stationary state). That is why the method is called the proportional-feedback one. So using the proportional feedback method can allow one to stabilize unstable stationary states

Now we will study the two dimensional case. Below, as it was underlined above in this paper we restrict ourselves to the isotropic case.

The system of equation (2) can be regarded as a mapping describing the time evolution of the velocity in the moving phase with $v_x(t+1)$ and $v_y(t+1)$ on the left-hand side and $v_x(t)$, $v_y(t)$ on the right-hand side.

In the case of isotropic distribution this mapping can be written as:

$$\begin{aligned} v_x(t+1) &= 1 - (1-c)\left(\frac{p}{2}v_y^{-1} + \frac{p}{2}(v_x^{-1} - 1)\right) = F_1, \\ v_y(t+1) &= 1 - (1-c)\left(\frac{p}{2}v_x^{-1} + \frac{p}{2}(v_y^{-1} - 1)\right) = F_2, \end{aligned} \quad (10)$$

where $p_x = p_y = \frac{p}{2}$.

The system of nonlinear mapping has two steady state solutions

$$v_{\pm} = \frac{1}{2}\left(1 + \frac{(1-c)p}{2} \pm \left(\left(1 + \frac{(1-c)p}{2}\right)^2 - 4(1-c)p\right)^{\frac{1}{2}}\right), \quad (11)$$

where $v = v_x = v_y$.

First of all we should find the condition of possible chaoticity in the system (10).

The stability of mapping is determined by the eigenvalues of the Jacobian matrix of the nonlinear mapping (10).

$$J = \frac{\partial(F_1, F_2)}{\partial(v_x(t), v_y(t))}, \quad (12)$$

It can be seen easily the eigenvalues of the Jacobian matrix is calculated by the following equation:

$$\lambda^2 - \lambda(1-c)\frac{p}{v^2} = 0, \quad (13)$$

From here we obtain easily that

$$\lambda_1 = 0,$$

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$$\lambda_2 = (1 - c) \frac{p}{v^2}, \quad (14)$$

In the last expression while calculating λ we use the steady state solutions (10). This simplification is justified at least for systems whose chaotic behavior has arisen out of stability of fixed points, see ref.15, also ref.20-21.

The mapping will exhibit chaotic behaviour, if the absolute values of λ exceed unity.

$$|\lambda_2| > 1, \quad (15)$$

. As the initial or original nonlinear mapping is symmetric over v_x and v_y considering only one of these variables as a driver will be sufficient. Take for the definiteness v_x variable as a driver. Then the replica subsystem (with the superscript "r") can be written as follows:

$$v_y^r(t+1) = 1 - (1 - c) \left(\frac{p}{2v_x(t)} + \frac{p}{2} \left(\frac{1}{v_y^r(t)} - 1 \right) \right) = H, \quad (16)$$

Then the Lyapunov exponent can be calculated as follows, ref.18

$$\Lambda = \ln \frac{\partial H}{\partial v_y^r} = \ln(1 - c) \frac{p}{2} \frac{1}{v^2}, \quad (17)$$

. For the chaos control (to be more specific for the synchronization of the evolution of the response system to the chaotic evolution of the initial nonlinear mapping when time goes to infinity) it is required that

$$\Lambda < 0, \quad (18)$$

. It will take place, if the following condition is satisfied:

$$(1 - c) \frac{p}{2v^2} < 1, \quad (19)$$

. Thus we have two inequalities: (19) and (15). If these inequalities do not contradict each other, then the replica approach allows us to perform chaos control. We see that chaos control within replica approach is realizable if

$$1 < (1 - c) p \frac{1}{v^2} < 2, \quad (20)$$

Although the restriction (20) is very severe in the sense that diapason of changing of traffic flow models parameters such as c , p could be very narrow. Nevertheless, as the analysis of the data presented in ref.8 shows that chaos synchronization would be possible within the replica approach. If this approach fails, we can apply nonreplica one to achieve our goal, as it was underlined above.

Now suppose that our attempts to perform chaos control failed within replica approach.

In this case we can try nonreplica approach. According to ref.18, within nonreplica approach we can use the following nonreplica response system(with the superscript " nr"):

$$\begin{aligned} v_x^{nr}(t+1) &= 1 - (1-c)\left(\frac{p}{2}(v_y^{nr})^{-1} + \frac{p}{2}(v_x^{-1} - 1)\right) + \alpha(v_x^{nr} - v_x) = F_3, \\ v_y^{nr}(t+1) &= 1 - (1-c)\left(\frac{p}{2}v_x^{-1} + \frac{p}{2}((v_y^{nr})^{-1} - 1)\right) + \beta(v_x^{nr} - v_x) = F_4, \end{aligned} \quad (21)$$

where α and β are the arbitrary constants.

Here again as in the previous case we consider v_x dynamical variable as the driver.

The Lyapunov exponents are the eigenvalues of the Jacobian:

$$J = \frac{\partial(F_3, F_4)}{\partial(v_x^{nr}(t), v_y^{nr}(t))}, \quad (22)$$

From (21) we easily establish that these exponents are solutions to the following equation:

$$\lambda^2 - \lambda(\alpha + (1-c)\frac{p}{2}v^{-2}) - \beta(1-c)\frac{p}{2}v^{-2} = 0, \quad (23)$$

Here v is the steady state solution of the original mapping. As it can be seen from (23) the roots of this equation λ_1 and λ_2 satisfy the relationships:

$$\begin{aligned} \lambda_1 + \lambda_2 &= \alpha + (1-c)\frac{p}{2}v^{-2}, \\ \lambda_1\lambda_2 &= -\beta(1-c)\frac{p}{2}v^{-2}, \end{aligned} \quad (24)$$

Remind that our aim is to satisfy the conditions $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Due to the arbitrariness of the constants α and β this can be done easily.

Up to now while performing chaos synchronization we have taken the advantage of using the presence of driving variables explicitly. Our calculations show that chaos synchronization is also reachable in case of parameter perturbation method, ref.22. Namely, we have shown that by changing the fraction of overpasses one can make the absolute values of $\lambda_{1,2}$ less than unity. Indeed, by assuming the following change for c :

$$c = c_1 - \alpha_c(v_y - v_{y_{ap}}), \quad (25)$$

(where: c_1 is the nominal value for the c in the original two dimensional model; α_c is the control coefficient to be found; v_y and $v_{y_{ap}}$ are response system and drive system orbits, respectively.) after lengthy calculations we obtain that for the isotropic case much sought eigenvalues are:

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= 2(1-c_1)\frac{p}{2}v_{ap}^{-2} + \frac{p}{2}\alpha_c(1-2v_{ap}^{-1}), \end{aligned} \quad (26)$$

Thus we have the real possibility to satisfy the condition for the chaos control: $|\lambda_2| < 1$.

Now let us make some estimations based on our approach. According to ref.8 for the fraction of overpasses $c = 0.5$ the critical value of car density when jam occurs

$$p_{cr} = \frac{6 - 32^{\frac{1}{2}}}{1 - c} = 0.686, \quad (27)$$

Then for $p > p_{cr}$ jam phase takes place with v close to zero. It means that for these values of c, p instability conditions hold. Now we will use the formulae (25) and (26) to resolve the jamming problem by changing c . Of course, while conducting calculations we should keep in mind that maximal values for c, v are unity and the jamming phase could be avoided by increasing c . From the condition $|\lambda_2| > 1$ (formula (26)) for the extreme case $v_{yap} = 1$ we obtain that $-1.9 < \alpha_c < 3.9$. Take for definiteness $\alpha_c = -1.5$. Further, having in mind that when synchronization takes place v_y is very close to v_{yap} , therefore assuming $v_y - v_{yap} = 0.1$ from the equation (25) we derive new value for the fraction of overpasses capable of resolving the traffic jamming: $c = 0.65$ which is very close to the value of $c, c = 0.657$ when there is no jamming at all; that is $p_{cr} = 1$, ref.8. Of course much depends on the degree of accuracy for synchronization between v_y and v_{yap} . But, in any case we can say safely that our results do not contradict to the conventional wisdom and gives the right direction of action.

In conclusion in this work we pointed out to the possibility of the stabilization of the unstable stationary states in one of one dimensional model with random delay. Also we have investigated the possibility of chaos control in one of two dimensional mapping in traffic flow within replica, nonreplica and parameter change approaches. As we indicated above nonreplica approach has advantages over the replica approach. One of them is the possibility to make Lyapunov exponents not only negative, but also larger in magnitude. This is very important from the application point of view, as the time required to achieve synchronization depends on the value of the largest Lyapunov exponent, ref.1-16, 18.

Speaking about the study of 2D CA traffic flow models one has to mention that although 2D CA models less representative (in comparison with 1D rule-184 CA) of real traffic flow, however they may be applicable to abstract traffic problems such as data packets in computer networks, ref.23. Besides, 2D CA models may be useful from the viewpoint of complex behavior in deterministic dynamics, ref.24-26.

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